Influences of the Lorentz symmetry violation on the interaction of the relativistic spin-zero particle with the Cornell-type non-minimal coupling

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05 May 2022

Abstract

Based on the Standard Model Extension, we study the influences of Lorentz symmetry breaking effects on the interaction of a neutral spin-zero Duffin–Kemmer–Petiau particle. This particle is non-minimal coupled to a Cornell-type potential, which represents a generalized relativistic quantum oscillator model. This system is considered in the background of the Lorentz symmetry breaking mechanics generated by an electric field and a fixed vector field, which in turn provides a type of magnetic dipole moment. The generalized Duffin–Kemmer–Petiau equation is constructed incorporating that non-minimal CPT-odd coupling. Exact analytical spectral properties are derived for two special configurations of the electric and fixed vector field, both exhibiting a cylindric symmetry.

Keywords: Lorentz symmetry violation; Generalized spin-zero Duffin–Kemmer–Petiau oscillator; Cornell-type non-minimal coupling; Inverse-square-type potential; Shifted radial oscillator harmonic potential.
PACS numbers: 04.20.Jb; 04.62.+v; 11.30.Qc; 12.60.-i; 13.40.Em.

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1 Introduction

The description of fundamental particles and their interactions can be achieved via a field theoretic approach following the well-known standard model (SM). This model incorporates the strong forces and electro-weak interactions and excellently reproduces phenomenological observations [1]. However, it fails in some fundamental issues such as the observed striking matter-antimatter imbalance [2]. In a way forward, by expanding the SM model, Kostelecký and Samuel [3] were the first to provide a possible scenario of the spontaneous breaking of Lorentz symmetry. Indeed, in the framework of string field theory, they stipulated spontaneous violation of Lorentz symmetry (LSV) generated by a four-vector field. In this way, an extension of the SM was formulated incorporating such LSV in the SM. Such so-called Standard Model Extensions (SME) have attracted much interest in the last decades, see for example ref. [4–13].

Hence, investigating the quantum properties of neutral or charged particles subject to the influence of background fields, which break the Lorentz symmetry as a possible scenario out of the SME, can be accomplished by applying the relevant non-minimal coupling corresponding to the background field in the non-relativistic and relativistic Hamiltonians [14–18]. In recent years, such LSV backgrounds have received a great deal of attention on many issues in various branches of physics in the non-relativistic and relativistic regimes [19–33]. For instance, the Rashba-type coupling induced by the LSV effect was studied in [34]. Investigations on the influence of planar Maxwell-Chern-Simons models endowed with a Lorentz-violating term on a fermionic system were done in [35, 36]. Let us also mention discussions on a kind of Landau-Aharonov-Casher quantization related to a relativistic fermion particle [37], on Dirac and Klein-Gordon (KG) particles in central potentials induced by the LSV effect [38–42], and discussions on a modified quantum electrodynamics model formed by usual minimally coupled QED with the addition of a non-minimal Lorentz-violating interaction [43]. Further examples are the construction of strong equivalence and Lorentz violating Lagrangians for QED [44] and the investigations on the influence of the LSV background on the relativistic Anandan quantum phase [45].

In 1989, Moshinsky and Szczepaniak [46] introduced the so-called Dirac oscillator, which represents such a non-minimal interaction scenario in the Dirac equation. This Dirac oscillator is indeed a quantum model for the relativistic harmonic oscillator corresponding to the spin-half fermionic field as in its non-relativistic limit the radial quadratic potential shows up. So far, Dirac oscillator models have been studied in the framework of quantum mechanics in various contexts such as the exact solutions of the two-dimensional Dirac oscillator [47], the examination of the behaviour of the Dirac oscillator in the Som–Raychaudhuri space-time [48], and the study of the interaction between the Dirac oscillator with gravitational fields generated by topological defects [49]. Other works to be mentioned are, the examination of the Dirac oscillator in the presence of Aharonov-Bohm and magnetic monopole potentials [50], the investigation of a fermion-antifermion pair interacting with an external uniform magnetic field in the presence of the Dirac oscillator coupling [51] and the study of the Dirac oscillator in a spinning cosmic string spacetime [52, 53].

The approach of Moshinsky and Szczepaniak has also been extended to other relativistic systems like the KG [54] and the spin-zero Duffin–Kemmer–Petiau (DKP) particle [55]. Such extensions also include a generalization of the linear potential representing the oscillator by a Cornell-type potential where, in addition to the linear term, a Coulomb-like term is present [56, 57].

The purpose of this work is to investigate the relativistic behavior of the neutral spin-zero Duffin–Kemmer–Petiau (DKP) particle, which interacts with a Cornell-type non-minimal coupling called the generalized DKP oscillator. This interaction is assumed to happen under the influence of a background induced by an electric field and a fixed space-like vector field resulting in a LSV scenario. We note that, to our knowledge, the generalized DKP oscillator has not yet been studied in a background of the LSV effect. Here we will considered two explicit scenarios of an electric field configuration and fixed space-like vector field representing a magnetic dipole moment. Both configuration will have a cylindirc symmetry and thus allow for explicit solutions of their spectral properties. It is worth mentioning that, in the context of relativistic quantum mechanics, the DKP equation is introduced as a first-order covariant equation compatible with the study of spin-zero and spin-one particles being electrically charged or not [58–61]. The study of bosons can be done with the DKP equation in a more comprehensive prescription than the KG and Proca equations because the DKP equation allows for a wider range of couplings not allowed by the KG and Proca equations [62–70].

This paper is organized as follows. In section 2, we start with the introduction of the Cornell-type non-minimal coupling and embed it in the DKP equation for a neutral spin-0 particle. We then introduce the LSV effect by adding an interaction of the neutral DKP particle with a priori arbitrary electromagnetic field via a fixed four-vector. The 5-component DKP equation for this scenario is constructed and reduced to an effective second order differential equation for the first component of the DKP spinor. This is achieved by considering two particular case, A and B, of field configurations. Both cases in essence result in a radial non-relativistic Schödinger-like equation for a two-dimension harmonic oscillator. These are then explicitly discussed in section 3, where we present the spectral properties of this DKP particle for both configurations accompanied by a detailed discussion of the effects of the various coupling parameters involved. We conclude the discussion in section 4 with a brief summary of our study.

2 The generalized DKP oscillator in a LSV background

Inspired by the work of Moshinsky and Szczepaniak on the Dirac oscillator model [46], more recently, a similar model has been entertained in the literature [56, 57] for the relativistic DKP particle, the so-called DKP oscillator. This model in essence incorporates a linear coupling term into the DKP equation. It is worth noting that despite incorporating such coupling into the DKP equation, it remains linear for both coordinates and momenta. This traditional non-minimal coupling is achieved by modifying the momentum operator as $p_\mu \to p_\mu + \mathrm{i}M\omega\eta^0\mathcal{F}_\mu$. Here $M > 0$ stands for the mass of the bosonic particle and the harmonic coupling is represented by the angular frequency $\omega > 0$. The four-vector $\mathcal{F}_\mu$ is given by $\mathcal{F}_\mu = (0, r, 0, 0)$, where r is the distance of the DKP particle from the z-axis. Let us note here that we will work with cylindrical space-coordinates throughout this paper. The 5×5-matrix η^0 is defined below in eq. (2.1) via the DKP matrix β^0.

The first objective of this work is to consider a generalization of this linear non-minimal coupling by substituting the non-vanishing component of $\mathcal{F}_\mu$ by the Cornell potential function $\mathcal{G}(r) = C_1 r + \frac{C_2}{r}$, that is, we now consider $\mathcal{F}_\mu = (0, \mathcal{G}(r), 0, 0)$ [71–73], which results in following modified non-minimal coupling to be incorporated in the DKP equation [57, 74]

$$p_\mu \to p_\mu + \mathrm{i}M\omega\eta^0 \left[C_1 r + \frac{C_2}{r}\right] \delta^r_\mu \,, \qquad \eta^0 = 2(\beta^0)^2 - 1\,. \tag{2.1}$$

In the above $C_1 \geq 0$ and $C_2 \in \mathbb{R}$ are constants, whereas β^0 represents the zero component of the standard spin-zero DKP matrices. Thus, introducing the modified non-minimal coupling (2.1) in the DKP equation gives rise to what we will now call the generalized DKP oscillator.

Our second objective is to consider this generalized DKP oscillator in the environment of a LSV background. That is, we intend to investigate the relativistic quantum dynamics in $1+3$-dimensional Minkowski space-time of a neutral DKP particle having spin zero in the presence of the LSV effect stemming from the non-minimal CPT-odd coupling to an external electromagnetic field.

Thus we are investigating the interaction of the neutral DKP particle with a Cornell-type non-minimal coupling in the background of the LSV effect in the relativistic limit using curvilinear coordinates. Therefore, we need to write the line element of Minkowski space-time in cylindrical coordinates as $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2$ and pursue to find the relativistic energy spectrum and associated wave functions for the generalized DKP oscillator in a background involving a axial-symmetric potential induced by the LSV effect. In the above and the remaining part of this paper we will work in natural units where $\hbar = 1 = c$. From a mathematical point of view, following the spinor theory, the behavior of the DKP spinor in the curvilinear coordinates is analogous to their behavior in curved space-time. As mentioned in ref. [37], in the local reference frame of an observer, the relevant spinor is described locally in the background of curved space-time. In this regard, based on a non-coordinate basis $\hat{\theta}^a = e^a{}_\mu(x)\mathrm{d}x^\mu$, one can construct the local frame corresponding to the observer such that the tetrads $e^a{}_\mu(x)$ constitute their components. Their inverse $e^\mu{}_a(x)$ obey the relation $\mathrm{d}x^\mu = e^\mu{}_a(x)\hat{\theta}^a$ and follow from the conditions $e^a{}_\mu(x)e^\mu{}_b(x) = \delta^a{}_b$ and $e^\mu{}_a(x)e^a{}_\nu(x) = \delta^\mu{}_\nu$. The metric tensor components and tetrads obey the condition $g_{\mu\nu}(x) = e^a{}_\mu(x)e^b{}_\nu(x)\eta_{ab}$, where the Minkowski metric tensor is denoted by $\eta_{ab} = \mathrm{diag}(-,+,+,+)$.

When studying the DKP equation in curved space-time, we must replace the partial derivative ∂_μ by the covariant derivative ∇_μ. The latter is given by $\nabla_\mu = \partial_\mu - \Gamma_\mu$ with the spinorial connection $\Gamma_\mu = -\mathrm{i}\omega_{\mu ab}\sigma^{ab}/2$. Here $\omega_{\mu ab}$ is the spin connection discussed further below and $\sigma^{ab} = \mathrm{i}\left[\beta^a, \beta^b\right]$, with β^a being the standard DKP matrices associated with flat space-time background [57]. Note that we use Latin indices $(a, b = 0, 1, 2, 3)$ for the local frame of an observer and for the space-time frame Greek indices $(\mu, \nu = t, r, \varphi, z)$ are used.

The DKP matrices introduces above obey the DKP algebra $\beta^a\beta^b\beta^c + \beta^c\beta^b\beta^a = \eta^{bc}\beta^a + \eta^{ba}\beta^c$. As is well-known the DKP algebra, in $1+3$-dimensional flat space-time, embraces three irreducible representations: a ten-dimensional representation corresponding to a spin-one boson, a five-dimensional representation related to spin-zero particles, which is our desired representation in this contribution, and the trivial one-dimensional representation. The standard DKP matrices β^a belonging to the five-dimensional representation can be expressed, see for example [60], as follows

$$\beta^0 = \begin{pmatrix} \theta & 0_{2\times3} \\ 0_{3\times2} & 0_{3\times3} \end{pmatrix}, \qquad \vec{\beta} = \begin{pmatrix} 0_{2\times2} & \vec{\tau} \\ -\vec{\tau}^{\mathrm{T}} & 0_{3\times3} \end{pmatrix}. \tag{2.2}$$

Here T denotes the transposition of a matrix and the non-vanishing sub-matrices are given by

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

In order to obtain the non-null components of the spinorial connection, we first need to find the components of the spin connection $\omega_{\mu ab}$. Accordingly, we can obtain the non-null components of the spin connections through the Maurer–Cartan structure equations, that is, $\mathrm{d}\hat{\theta}^a + \omega^a{}_b \wedge \hat{\theta}^b = 0$, with $\omega^a{}_b = \omega_\mu{}^a{}_b\,\mathrm{d}x^\mu$, which is written in the absence of torsion. Hence, based on the two non-vanishing components of the spin connections $\eta_{22}\,\omega_\varphi{}^2{}_1 = -\eta_{11}\,\omega_\varphi{}^1{}_2 = 1$, we arive at the only non-vanishing component of the spinorial connections

$$\Gamma_\varphi = \begin{pmatrix} 0_{3\times3} & -\tau_3^{\mathrm{T}} \\ \tau_3 & 0_{2\times2} \end{pmatrix}. \tag{2.4}$$

In a next step we are now setting up the full model we are interested in. That is, we consider the generalised DKP oscillator being represented by the non-minimal coupling proposed in eq. (2.1). This neutral spin-zero DKP particle, in addition, interacts with an external electromagnetic field via an LSV interaction. Hence, our starting point is the following non-minimal coupling in curvilinear coordinates

$$\mathrm{i}\beta^\mu\nabla_\mu \to \mathrm{i}\beta^\mu\partial_\mu + \mathrm{i}\beta^r M\omega\eta^0\left[C_1 r + \frac{C_2}{r}\right] - i\beta^\varphi\Gamma_\varphi - g\mathrm{b}^\nu\tilde{F}_{\mu\nu}(x)\beta^\mu. \tag{2.5}$$

In the above β^μ are the generalized DKP matrices given via $\beta^\mu = e^\mu{}_a(x)\beta^a$ in terms of the standard DKP matrices via the inverse tetrads. Explicitly, we have $\beta^t = \beta^0$, $\beta^r = \beta^1$, $\beta^\varphi = \beta^2/r$ and $\beta^z = \beta^3$. Furthermore, in (2.5) the real parameter g characterizes the coupling between the fixed four-vector b^ν

being the origin of the LSV and the dual electromagnetic tensor $\tilde{F}_{\mu\nu}(x) = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}(x)$. Here the electromagnetic field tensor is given by $F^{\alpha\beta}(x)$ whose components represent the electric field vector E_i and magnetic field vector B_i in the usual way, i.e. $F^{0i} = -F^{i0} = E^i$ and $F^{ij} = \epsilon^{ijk}B_k$, with $i, j = 1, 2, 3$.

Inspired by the admirable paper of Lunardi et al. [75], let us mention that for free fields, that is, in the absence of the electromagnetic fields in the relevant framework, there is a perfect equivalence between the DKP equation and KG and Proca equations. But in the presence of the interaction associated with the electromagnetic field incorporated in the DKP equation via a non-minimal coupling, doubts about this equivalence come up. It should be noted that, in this work, the LSV background is described by the introduction of a non-minimal coupling in the DKP equation given by eq. (2.5).

Let us now elaborate a little more on the term related to the LSV effect, which is the last term of the non-minimal coupling given by eq. (2.5). In the literature, the fixed four-vector b^ν is supposed to represent a vector field that violates the Lorentz symmetry. The constant g together with the spatial part of the fixed four-vector b^ν gives rise to a type of magnetic dipole moment associated with a neutral DKP particle in the background of the LSV effect, that is, $\vec{\mu} = g\vec{\mathrm{b}}$ [16, 17].

Independently of the particle's spin, intervening the Lorentz-symmetry violating background vector in spacetime may give rise to some new properties to the particles, such as kinds of dipole moment and quantum phase. In fact, this is the role that the last term of eq. (2.5) can play. Meanwhile, the spinor of the DKP equation has been defined in the local reference frame of the observers. With these descriptions, the equivalence between the DKP equation and KG does not exist in this work due to applying the corresponding non-minimal coupling and the present DKP wave function Ψ. In this case, the contradiction between the equations DKP and KG disappears if the physical form of the DKP field is modified and written in accordance with our desired non-minimal coupling.

Finally, let us mention that this interaction term can be written a bit more explicit as follows

$$-g\mathrm{b}^\nu \tilde{F}_{\mu\nu}(x)\beta^\mu \equiv -\beta^t g\,\vec{\mathrm{b}}\cdot\vec{B} + \vec{\beta}\cdot\left(g\,\mathrm{b}^0\vec{B} - g\left(\vec{\mathrm{b}}\times\vec{E}\right)\right), \tag{2.6}$$

in which the vector $\vec{\beta}$ is given by $\vec{\beta} = (\beta^r, \beta^\varphi, \beta^z)$, the fixed time-like vector is denoted by b^0 and the fixed space-like vector is represented by $\vec{\mathrm{b}} = (\mathrm{b}^1, \mathrm{b}^2, \mathrm{b}^3)$. Obviously, this gives rise to an effective vector potential in the form of $\vec{A}^{\mathrm{eff}} = g\mathrm{b}^0\vec{B} - g(\vec{\mathrm{b}}\times\vec{E})$, which in turn results in an effective magnetic $\vec{B}^{\mathrm{eff}} = \vec{\nabla}\times\vec{A}^{\mathrm{eff}}$ in the LSV background. Therefore, the expression $\vec{A}^{\mathrm{eff}} - g\,\vec{\mathrm{b}}\cdot\vec{B}$ can provide the Anandan quantum phase for a corresponding neutral particle if the appropriate electric and magnetic fields are considered. Note that this geometric phase is considered as an Abelian quantum phase under the background of the LSV effect defined by a fixed vector field.

In going forward we will restrict ourselves to a vanishing magnetic field and investigate two possible scenarios of the LSV effect by introducing two different cases of an electric field together with a special choice of the spatial part of the fixed four-vector field in the following way.

$$\text{Case A}: \qquad \vec{\mathrm{b}} = (0, 0, \mathrm{b}^3), \qquad \vec{E} = (E_r, 0, 0), \tag{2.7a}$$

$$\text{Case B}: \qquad \vec{\mathrm{b}} = (\mathrm{b}^1, 0, 0), \qquad \vec{E} = (0, 0, E_z). \tag{2.7b}$$

With that choice it is obvious that the above interaction term (2.6) is reduced to a single φ-component of the form $-g(\vec{\mathrm{b}}\times\vec{E})_\varphi$.

Under these assumptions the wave equation for the generalized DKP oscillator following from the highly non-trivial non-minimal coupling schema (2.5) simplifies to

$$\begin{aligned}&\left[\mathrm{i}\beta^0\partial_t + \mathrm{i}\beta^1\left(\partial_r + M\omega\eta^0\left[C_1 r + \frac{C_2}{r}\right]\right) + i\frac{\beta^2}{r}\left(\partial_\varphi - \Gamma_\varphi + \mathrm{i}g(\vec{\mathrm{b}}\times\vec{E})_\varphi\right)\right.\\ &\left.+\mathrm{i}\beta^3\partial_z - M\right]\Psi\left(t, \vec{r}\right) = 0.\end{aligned} \tag{2.8}$$

Given the existence of time-independent interactions in this approach and the presence of a cylindrical symmetry, one can decompose eq. (2.8) into a set of time-independent equations through the following ansatz of the spin-zero DKP spinor

$$\begin{aligned}&\Psi\left(t,\vec{r}\right)=\mathrm{e}^{-\mathrm{i}\mathcal{E}t+\mathrm{i}m\varphi+\mathrm{i}kz}\psi\left(r\right),\\&\psi\left(r\right)=\left(\psi_1\left(r\right),\psi_2\left(r\right),\psi_3\left(r\right),\psi_4\left(r\right),\psi_5\left(r\right)\right)^{\mathrm{T}}.\end{aligned}\tag{2.9}$$

Here the energy of the scalar boson in this background is indicated by $\mathcal{E}$, the eigenvalues of the z-component of the angular momentum operator $\hat{L}_z=-\mathrm{i}\partial_\varphi$ and linear momentum operator $\hat{p}_z=-\mathrm{i}\partial_z$ are represented by $m\in\mathbb{Z}$ and $k\in\mathbb{R}$, respectively. With above ansatz (2.9) eq. (2.8) results in the following five coupled equations for the components of the spin-zero DKP spinor:

$$\begin{aligned}&-M\psi_1(r)+\mathcal{E}\psi_2(r)+\left(-\mathrm{i}\partial_r-\frac{\mathrm{i}}{r}+\mathrm{i}M\omega\left[C_1r+\frac{C_2}{r}\right]\right)\psi_3(r)\\&\quad+\frac{1}{r}\left(m+g(\vec{\mathrm{b}}\times\vec{E})_\varphi\right)\psi_4(r)+k\psi_5(r)=0,\end{aligned}\tag{2.10a}$$

$$\mathcal{E}\psi_1(r)-M\psi_2(r)=0,\tag{2.10b}$$

$$\left(\mathrm{i}\partial_r+\mathrm{i}M\omega\left[C_1r+\frac{C_2}{r}\right]\right)\psi_1(r)-M\psi_3(r)=0,\tag{2.10c}$$

$$-\frac{1}{r}\left(m+g(\vec{\mathrm{b}}\times\vec{E})_\varphi\right)\psi_1(r)-M\psi_4(r)=0,\tag{2.10d}$$

$$-k\psi_1(r)-M\psi_5(r)=0.\tag{2.10e}$$

Elimination of four components results in a second order differential equation for, say, the first component explicitly given by

$$\begin{aligned}&\frac{\mathrm{d}^2\psi_1(r)}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\psi_1(r)}{\mathrm{d}r}+\Big[\mathcal{E}^2-M^2-k^2+2M\omega C_1-2C_1C_2M^2\omega^2\\&-M^2\omega^2C_1^2r^2-\frac{1}{r^2}\left(M^2\omega^2C_2^2+m^2+2mg\,(\vec{\mathrm{b}}\times\vec{E})_\varphi+\left(g\,(\vec{\mathrm{b}}\times\vec{E})_\varphi\right)^2\right)\Big]\,\psi_1(r)=0.\end{aligned}\tag{2.11}$$

This equation is identical in form with the non-relativistic radial Schödinger eigenvalue equation of a two-dimension harmonic oscillator with frequency ωC_1 and a modified angular momentum and energy. Actually this comment is only true for $(\vec{\mathrm{b}}\times\vec{E})_\varphi$ being a constant. However, we may also allow for $(\vec{\mathrm{b}}\times\vec{E})_\varphi\sim r^2$, which in essence is another two-dimension harmonic oscillator with different frequency. Hence, in both cases its explicit solution can easily be obtained. This we will discussed in detail in the next section for the two special cases related to the electric field and the fixed space-like vector field as specified in (2.7). In doing so we are able to explicitly investigate the influences of the LSV background on these two scenarios within our approach.

3 Exact solutions under the influences of two LSV scenarios

As anticipated above, in this section we will present explicit results for the two special cases of the electric field and the fixed space-like vector field proposed in eq. (2.7). With the radial second-order differential equation (2.11) being in essence the radial problem of an isotropic harmonic oscillator, which in turn can be reduce to the well-studied Whittaker equation, we will be able to present closed form solutions in both cases. Let us note that this kind of problem where two particular cases reduce to an harmonic oscillator problem was also found when investigating the interaction of a magnetic quadrupole moment of a moving particle in an elastic medium with a rotating frame in the presence of a screw dislocation [76].

3.1 Case A

This case belongs to a uniform electric field in the radial direction, that is orthogonal to the fixed vector field which is aligned with the z-axis. In accordance with (2.7a) they are given by

$$\vec{\mathrm{b}} = \mathrm{b}^3\,\hat{z}, \qquad \vec{E} = E_0\,\hat{r}, \tag{3.1}$$

with $\mathrm{b}^3 > 0$ and $E_0 \in \mathbb{R}$ being arbitrary but constant parameters.

Now let us analyze the relativistic behavior of the neutral spin-zero DKP particle interacting with a Cornell-type non-minimal coupling under the background involving a type of central potential induced by an LSV scenario generated by above choice of electric field and fixed space-like vector. To simplify this discussion let us put eq. (2.11) for the case at hand into a radial-harmonic-oscillator-like form, that is,

$$-\frac{1}{2M}\left(\frac{\mathrm{d}^2\psi_1(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi_1(r)}{\mathrm{d}r}\right) + \left[\frac{\ell_A^2}{2Mr^2} + \frac{M}{2}\Omega_A^2 r^2\right]\psi_1(r) = \mathsf{E}_A\psi_1(r)\,. \tag{3.2}$$

Here we have introduced the quantities

$$\mathsf{E}^A = \frac{1}{2M}\left(\mathcal{E}^2 - M^2 - k^2 + 2M\omega C_1 - 2C_1C_2M^2\omega^2\right), \tag{3.3a}$$

$$\ell_A^2 = m^2 + M^2\omega^2 C_2^2 + 2mg\mathrm{b}^3E_0 + \left(g\mathrm{b}^3E_0\right)^2, \tag{3.3b}$$

$$\Omega_A = \omega C_1, \tag{3.3c}$$

which in essence represent the energy, angular momentum and frequency of the two-dimensional oscillator, respectively. Note that we assume here $\ell_A \geq 0$.

In this form, eq. (3.2) allows us to discuss the effect of the uniform electric field E_0 on the generalized DKP oscillator. Indeed, it becomes obvious that the induced magnetic dipole moment corresponding to a neutral DKP particle, $\mu = g\mathrm{b}^3$, interacts with the electric field. To be more precise, this interaction results in an inverse-square-type potential stemming from the LSV scenario determined by eq. (3.1) in the form of

$$\mathcal{V}_A(r) = \frac{2mg\mu E_0 + (g\mu E_0)^2}{2Mr^2}. \tag{3.4}$$

If, in addition, we consider the centrifugal part represented by the m^2-term in (3.3b) we observe that

$$\mathcal{V}_A(r) + \frac{m^2}{2Mr^2} = \frac{(m + g\mu E_0)^2}{2Mr^2}. \tag{3.5}$$

That is, the dipole moment in the presence of the constant electric field induces a shift in the angular momentum quantum number $m \to m + g\mu E_0$, which is similar to the shift induced by the Aharanov-Bohm setup of a charged particle encircling a magnetic flux. Let us finally point out that the generalisation of the DKP oscillator represented by the constant C_2 adds another contribution to the $1/r^2$-potential in (3.2) but does not influence the frequency (3.3c) of the oscillator. The constant shift in the energy (3.3a) only depends on the parameters C_1 and C_2 characterizing the generalized DKP oscillator.

To conclude our discussion on case A, let us recall the well-known energy eigenvalues and radial eigenfunctions of the 2-dim. harmonic oscillator, which read

$$\begin{aligned} \mathsf{E}^A_{nm} &= \Omega_A\left(2n + \ell_A + 1\right), \qquad n \in \mathbb{N}_0 \\ \psi^A_{1\,nm}(r) &= \sqrt{\frac{2M\Omega_A\Gamma(n+1)}{\Gamma(n+\ell_A+1)}}\left(M\Omega_a r^2\right)^{\ell_A/2}\mathrm{e}^{-\frac{M\Omega_A}{2}r^2}\,\mathcal{L}_n^{\ell_A}(M\Omega_A r^2). \end{aligned} \tag{3.6}$$

In the above $\mathcal{L}_n^{\ell_A}$ stands for the associated Laguerre polynomial of degree n. The dependency on the angular momentum quantum $m \in \mathbb{Z}$ is encoded in ℓ_A via (3.3b). The normalization of (3.6) is with respect to the Lebesque measure $r\mathrm{d}r$ on the positive real line. The dependency on quantum number $k \in \mathbb{R}$ is trivially contained in E_{nm}^A via (3.3a) and will not be explicitly indicated by a subscript to keep notation simple.

Having found the first component of the radial DKP spinor the others are given via eqs. (2.10) and explicitly read

$$\psi_{2\ nm}^A(r) = \frac{\mathcal{E}_{nm}^A}{M}\psi_{1\ nm}^A(r), \tag{3.7a}$$

$$\psi_{3\ nm}^A(r) = \frac{1}{M}\left(\mathrm{i}\partial_r + \mathrm{i}M\omega\left[C_1 r + \frac{C_2}{r}\right]\right)\psi_{1\ nm}^A(r), \tag{3.7b}$$

$$\psi_{4\ nm}^A(r) = -\frac{1}{M}\left[\frac{m}{r} + \frac{g\mathrm{b}^3 E_0}{r}\right]\psi_{1\ nm}^A(r), \tag{3.7c}$$

$$\psi_{5\ nm}^A(r) = -\frac{k}{M}\psi_{1\ nm}^A(r). \tag{3.7d}$$

These in turn provide us with via eq. (2.9) with the complete DKP spinor wave functions corresponding to the case A scenario in this approach in the form of

$$\Psi_{nm}^A(t,\vec{r}) = \mathrm{e}^{-\mathrm{i}\mathcal{E}_{nm}^A t + \mathrm{i}m\varphi + \mathrm{i}kz}\begin{pmatrix} \psi_{1\ nm}^A(r) \\ \frac{\mathcal{E}_{nm}^A}{M}\psi_{1\ nm}^A(r) \\ \frac{1}{M}\left(\mathrm{i}\partial_r + \mathrm{i}M\omega\left[C_1 r + \frac{C_2}{r}\right]\right)\psi_{1\ nm}^A(r)v \\ -\frac{1}{M}\left[\frac{m}{r} + \frac{g\mathrm{b}^3 E_0}{r}\right]\psi_{1\ nm}^A(r) \\ -\frac{k}{M}\psi_{1\ nm}^A(r) \end{pmatrix}. \tag{3.8}$$

The above spinor represents a stationary state associated with the relativistic energy eigenvalues given via eqs. (3.3a) and (3.6) by

$$\mathcal{E}_{nm}^A(k) = \pm\left[k^2 + M^2 + 2M^2\omega^2 C_1 C_2 + 2M\omega C_1\left(2n + \sqrt{(M\omega C_2)^2 + (m + g\mathrm{b}^3 E_0)^2}\right)\right]^{1/2}. \tag{3.9}$$

Obviously, being a relativistic system we have to consider both signs in eq. (3.9). In fact, physically, it means that the provided energy eigenvalues with positive and negative signs correspond to particles and antiparticles, respectively. Let us note that this energy spectrum is symmetric about zero with a non-vanishing gap given by the lowest positive and largest negative eigenvalues for $n = 0 = k$,

$$\mathcal{E}_{0m_0}^A(0) = \pm\left[M^2 + 2M\omega C_1\sqrt{(M\omega C_2)^2 + (m_0 + g\mathrm{b}^3 E_0)^2} + 2M^2\omega^2 C_1 C_2\right]^{1/2}. \tag{3.10}$$

In the above m_0 is the integer number m which minimises the quadratic form $(m + g\mathrm{b}^3 E_0)^2$. If we assume that the quantity $g\mathrm{b}^3 E_0$ takes on an integer value, say $-m_0$, the effect of the dipole-electric field interaction disappears as it is fully absorbed in the replacement $m \to m + m_0$. This situation is similar to the effect found in an Aharonov-Bohm setup when the enclosed magnetic flux is an integer multiple of the flux quantum. In that case the energy gap (3.10) reduces to

$$\mathcal{E}_{0m_0}^A(0) = \pm\left[M^2 + 2M^2\omega^2 C_1(|C_2| + C_2)\right]^{1/2}, \tag{3.11}$$

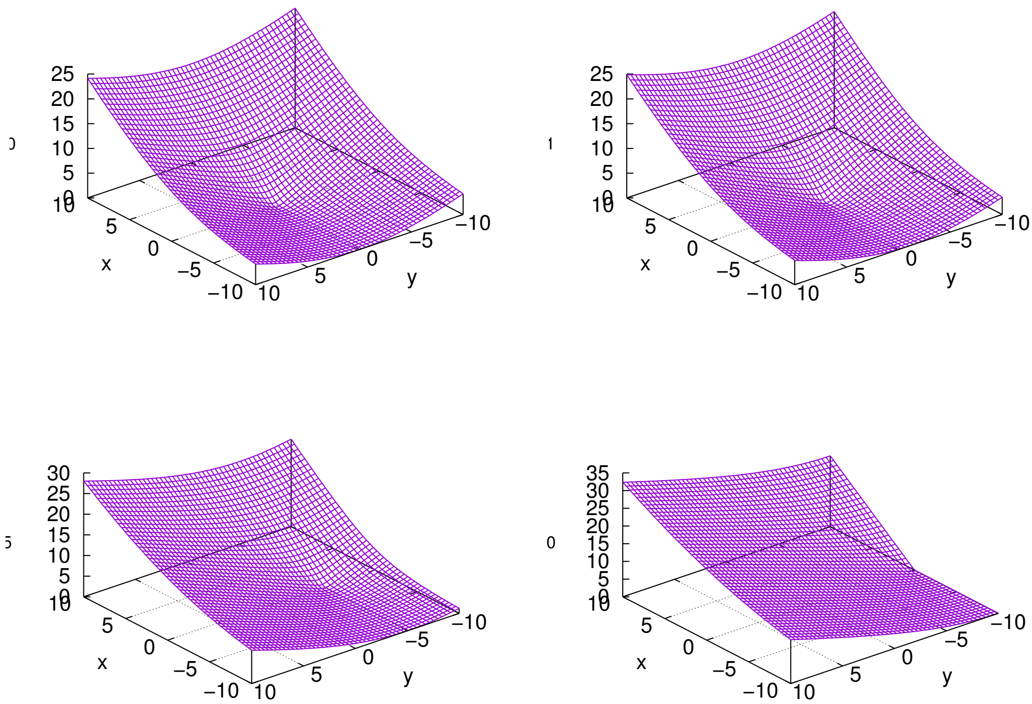

Figure 1: The squared energy gap Δ_m^A as defined in (3.12) for $m = 0, 1, 5$ and 10 over the dimensionless parameters $x = M\omega C_2$ and $y = g\mathrm{b}^3 E_0$.

which for $C_2 < 0$ simplifies to that of the free DKP particle $\mathcal{E}_{0m_0}^A = \pm M$ and no longer depends on the DKP oscillator parameters. However, for $C_2 > 0$ the gap is increased to $M^2 + 4M^2\omega^2 C_1 C_2$. In order to visualize this behavior let us consider the dimensionless quantity

$$\Delta_m^A = \frac{\left[\mathcal{E}_{0m}^A(0)\right]^2 - M^2}{2M\omega C_1} = M\omega C_2 + \sqrt{(M\omega C_2)^2 + (m + g\mathrm{b}^3 E_0)^2}\,, \tag{3.12}$$

which is a kind of squared energy gap on top of the usual free particle gap M^2. It also represents, for a fixed m, the ground-state energy of the system in the non-relativistic limit. Note that for large M we may set $|\mathcal{E}_{0m}^A(0)| \simeq M + E_{NR}^A$ and hence we have $\left[\mathcal{E}_{0m}^A(0)\right]^2 - M^2 \simeq 2ME_{NR}^A$, which implies $E_{NR}^A = \omega C_1 \Delta_m^A$. In figure 1 we have plotted the quantity Δ_m^A for the values $m = 0, 1, 5$ and 10 against the parameters $x = M\omega C_2$ and $y = g\mathrm{b}^3 E_0$.

To conclude case A, the provided exact analytical spectral properties in eqs. (3.8) and (3.9) describe the relativistic behavior of an oscillating system corresponding to the neutral spin-zero DKP particle interacting with the uniform cylinder-symmetric electric field via the induced magnetic dipole moment defined by eq. (3.1) in the background of the Lorentz symmetry breaking. We also note that the background of the LSV effect, arising from considering the uniform electric field and the fixed space-like vector given by eq. (3.1), becomes invisible when it is quantised, i.e., $g\mathrm{b}^3 E_0 \in \mathbb{Z}$

3.2 Case B

Now we turn our focus on the second case anticipated in (2.7). As briefly pointed out at the end of section 2, there are two choices available for the electric field to result in an effective two-dimensions harmonic oscillator problem. The choice for a constant electric field was taken in the previous case. Here we will look at the second choice where the electric field increases quadratically with the distance from the z-axis. To be more explicit we will choose the electric field configuration in the direction of the z-axis and the fixed space-like vector in the radial direction as follows.

$$\vec{\mathrm{b}} = \mathrm{b}^1\,\hat{r}, \qquad \vec{E} = \frac{\lambda}{2}\,r^2\,\hat{z}. \tag{3.13}$$

As before $\mathrm{b}^1 \geq 0$ is a non-negative constant. The electric field is such that it points into the positive ($\lambda > 0$) or negative ($\lambda < 0$) z-direction and its strength depends quadratically on the distance to that axis. Obviously, we allow for a real parameter $\lambda \in \mathbb{R}$.

In analogy to the previous case we can now study the relativistic behavior of the spin-zero DKP particle related to such an oscillating system in the presence of the electric field configuration and fixed space-like vector as given in (3.13). Again we end up with the radial Schödinger-type equation of a two-dimensional harmonic oscillator

$$-\frac{1}{2M}\left(\frac{\mathrm{d}^2\psi_1(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi_1(r)}{\mathrm{d}r}\right) + \left[\frac{\ell_B^2}{2Mr^2} + \frac{M}{2}\Omega_B^2 r^2\right]\psi_1(r) = \mathsf{E}^B\psi_1(r)\,. \tag{3.14}$$

However, now the energy, angular momentum quantum number and frequency are, respectively, given by

$$\mathsf{E}^B = \frac{1}{2M}\left(\mathcal{E}^2 - M^2 - k^2 + 2M\omega C_1 - 2C_1C_2M^2\omega^2 + mg\mathrm{b}^1\lambda\right), \tag{3.15a}$$

$$\ell_B^2 = m^2 + M^2\omega^2 C_2^2, \tag{3.15b}$$

$$\Omega_B^2 = \omega^2 C_1^2 + \left(g\mathrm{b}^1\frac{\lambda}{2M}\right)^2. \tag{3.15c}$$

At this stage let us note that in contrast to scenario A, here the electric field configuration does not influence the effective angular momentum parameter (3.15b). However, it does contribute to the energy parameter (3.15a) by a m-dependent shift and also increases the harmonic oscillator frequency. Its total effect is represented by a shifted radial oscillator harmonic potential as

$$\mathcal{V}_B(r) = \frac{mg\mathrm{b}^1}{2M}\lambda + \frac{M}{2}\left(g\mathrm{b}^1\frac{\lambda}{2M}\right)^2 r^2. \tag{3.16}$$

Whereas the harmonic oscillator part can simply be absorbed by redefining the frequency Ω_B, some more care might be required due to the m-dependent shift. This can be absorbed in the energy E^B as long as m is fixed. However, as $m \in \mathbb{Z}$ it is neither bounded from below nor from above. Therefore, we may need to look into the behavior of the system under consideration for large $m \to \pm\infty$. This we will done after presenting the explicit results.

The explicit solution of (3.14) is identical in form with that for case A by replacing the parameters (3.3) with index A by the parameters (3.15) with index B. That is, we have

$$\begin{aligned} \mathsf{E}^B_{nm} &= \Omega_B\left(2n + \ell_B + 1\right), \qquad n \in \mathbb{N}_0 \\ \psi^B_{1\,nm}(r) &= \sqrt{\frac{2M\Omega_B\Gamma(n+1)}{\Gamma(n+\ell_B+1)}}\,\left(M\Omega_B r^2\right)^{\ell_B/2}\mathrm{e}^{-\frac{M\Omega_B}{2}r^2}\,\mathcal{L}_n^{\ell_B}(M\Omega_B r^2), \end{aligned} \tag{3.17}$$

which in turn leads us to the primary wave function for the current case,

$$\Psi^B_{nm}(t,\vec{r}) = \mathrm{e}^{-\mathrm{i}\,\mathcal{E}^B_{nm}t + \mathrm{i}m\varphi + \mathrm{i}kz}\begin{pmatrix} \psi^B_{1\,nm}(r) \\ \frac{\mathcal{E}^B_{nm}}{M}\,\psi^B_{1\,nm}(r) \\ \frac{1}{M}\left(\mathrm{i}\partial_r + \mathrm{i}M\omega\left[C_1 r + \frac{C_2}{r}\right]\right)\psi^B_{1\,nm}(r) \\ -\frac{1}{M}\left[\frac{m}{r} - g\mathrm{b}^1\frac{\lambda}{2}r\right]\psi^B_{1\,nm}(r) \\ -\frac{k}{M}\,\psi^B_{1\,nm}(r) \end{pmatrix}. \tag{3.18}$$

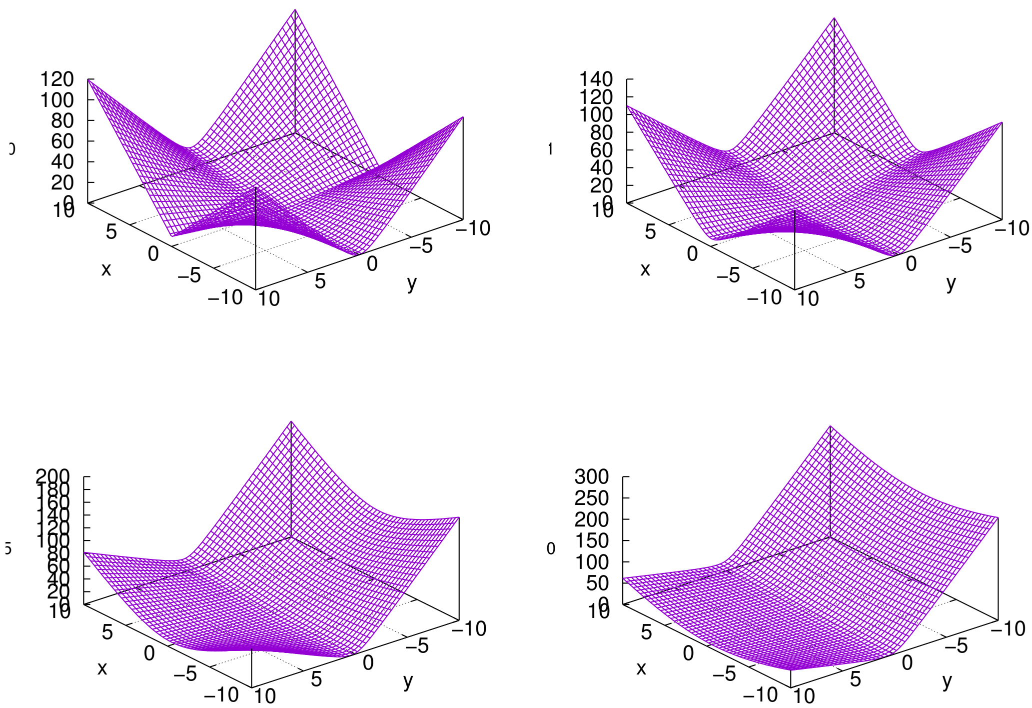

Figure 2: The energy gap Δ_m^B as defined in (3.20) for the parameters $m = 0, 1, 5$ and 10 over the dimensionless parameters $x = M\omega C_2$ and $y = g\mathrm{b}^1\lambda/2M\omega C_1$. Note that Δ_m^B is invariant under the replacement $(y, m) \to (-y, -m)$.

The associated relativistic energy eigenvalues can also be given in closed form as follows

$$\begin{aligned}\mathcal{E}_{nm}^B(k) = \pm \Big[&k^2 + M^2 - 2M\omega C_1 + 2M^2\omega^2 C_1 C_2 - mg\mathrm{b}^1\lambda \\ &+2M\left(2n+1+\left[M^2\omega^2C_2^2+m^2\right]^{1/2}\right)\left(\omega^2C_1^2+\left(g\mathrm{b}^1\frac{\lambda}{2M}\right)^2\right)^{1/2}\Big]^{1/2}.\end{aligned} \tag{3.19}$$

As anticipated above we need to take a closer look on the result for large $|m| \to \infty$. For doing so let us consider, as in case A, the dimensionless quantity

$$\begin{aligned}\Delta_m^B &= \frac{\left[\mathcal{E}_{0m}^B(0)\right]^2 - M^2}{2M\omega C_1} \\ &= -1 + M\omega C_2 - m\frac{g\mathrm{b}^1\lambda}{2M\omega C_1} + \left(1+\left[M^2\omega^2C_2^2+m^2\right]^{1/2}\right)\left(1+\left(\frac{g\mathrm{b}^1\lambda}{2M\omega C_1}\right)^2\right)^{1/2},\end{aligned} \tag{3.20}$$

which, again, characterizes the energy gap between the positive and negative parts of the energy spectrum as well as the ground-state energy in the non-relativistic limit. In figure 2 we have plotted the quantity (3.20) for $m = 0, 1, 5$ and 10 against the two dimensionless parameters $x = M\omega C_2$ and $y = g\mathrm{b}^1\lambda/2M\omega C_1$. These figures clearly indicate that $\Delta_m^B \geq 0$ and hence we conclude that the energy gap is bounded from below for all vlaues of parameters involved by $|\mathcal{E}_{nm}^B(k)| \geq M$.

With above solutions, it is possible to discuss the relativistic behavior of the neutral spin-zero DKP particle under the background having the shifted radial oscillator harmonic potential induced by the LSV scenario, which is defined by the adopted fixed space-like vector in the presence of the considered electric field configuration in the current case. From eq. (3.19), one can explicitly see the effects induced by the magnetic dipole moment related to the spin-zero DKP particle in the presence of the Cornell-type non-minimal coupling interacting with the radial electric field.

4 Conclusions

In this paper we have studied the spectral properties of a generalized DKP oscillator for spin-0 in the background of an external electromagnetic interaction generating a LSV effect. Two specific external field configurations have been considered. Both cases could be reduced to the radial Schödinger problem of a two-dimensional harmonic oscillator allowing for an explicit solution of the DKP eigenvalue problem.

In the first case discussed, case A, we found that the LSV has a minimal effect in shifting the angular momentum and becomes invisible in case where the coupling parameters obey the quantisation condition $g\mathrm{b}^3 E_0 \in \mathbb{Z}$. The spectral gap between positive and negative energy eigenvalues, in essence, depends only on the parameters C_1 and C_2 of the Cornell-type potential, see eq. (3.11), and this dependency even disappears completely for $C_2 > 0$, in which case the gap is simple that of the free DKP particle.

The situation is different in case B, where this energy gap significantly depends not only on the Cornell parameters C_1 and C_2 but also on the value $g\mathrm{b}^1\lambda$ characterizing the interaction with the electric field. This was clearly shown in the plots presented in figure 1. The energy gap appears, for all values of parameters, to be bounded by that of the free DKP particle.

To summarize, for both cases we were able to present the exact analytical spectral properties, that is, the energy eigenvalues and eigenfunctions, and their dependencies on the various parameters of the models under investigation. Due to the underlying cylindrical symmetry the obtained eigenvalues and eigenfunctions are presented in terms of the three quantum numbers $k \in \mathbb{R}$, $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, which stand for the wave number of the free motion along the z-axis, the angular momentum for the motion around that axis and for the radial quantum number, respectively. The dependency of the spectral properties on the various coupling constants was also given explicitly. These are $\omega > 0$, $C_1 > 0$ and $C_2 \in \mathbb{R}$ for the Cornell-type interaction, and $E_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ for the strength of the electric field in case A and B, respectively. In addition, in both cases, the interaction between the magnetic dipole moment b and the electric field was explicated.

Acknowledgement

The authors thank the referee for a thorough reading of our manuscript and constructive suggestions.

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